JKAU: Sci., **vol. 11,** pp. 55-62 (1419 A.H. / 1999 A.D.)

On the Primitive Classes of $K_*(BU)$

KHALED S. FELALI
Department of Mathematical Sciences,
Umm Al-Qura University, Makkah Al-Mukarramah, Saudi Arabia

ABSTRACT. We give a characterization of the primitive classes of $K_*(BU(n))$ in terms of its rational generators and use that to determine new primitive classes.

Introduction

The *BU*-ring spectrum *K* determines a generalized homology theory K_* with coefficient group

$$K_n(pt) = [S^n , K] = \pi_n(K)$$

It is well known that $K_*(K)$ can be regarded as a Hopf algebra over $\pi_*(K)$, and for each *X* (space or spectrum) we can define a coaction map

$$\psi : K_*(X) \to K_*(K) \otimes_{\pi*(K)} K_*(X)$$

which gives $K_*(K)$ the structure of a comodule over $K_*(K)$. The primitive submodule $PK_*(X)$ is defined by

$$PK_*(X) = \{ \alpha \in K_*(X) \colon \psi(\alpha) = 1 \otimes \alpha \}$$

In [1] we studied the case $X = MU(2)$ and gave a characteristic theorem which determines the primitive classes of $K_*(BU)(2))$ in terms of its rational generators. In [2] the same case is studied (among other things) where $K_*(BU(2))$ is identified with a certain submobule of Q[*X*, *Y*] whose homogeneous elements determine the primitive classes of $K_*(BU(2))$. In [3] the authors generalized the case of [2] to the space *BU*[*n*] where they identify $K_*(BU(n))$ with a certain submodule of Q[x_1 , x_2, ... x_n] whose homogeneous elements again determine the primitive classes of $K_*(BU(n))$.

Although the above paper can be regarded as the leading and most comprehensive study concern *BU*(*n*), but the primitive classes which were constructed there are all derived initially from *BU*(2).

Here we generalize the results of [1] and give a characterization of the primitive classes of $K_*(BU(n))$ in terms of its rational generators and use that to determine new primitive classes of $K_*(BU(n))$ which are not derived from $K_*(BU(2))$.

§ 1. Notations

Let $\{ \beta_1 , \beta_2, \ldots , \beta_n , \ldots \}$ be the usual $\pi_*(K)$-basis of $K_*(CP^\infty)$. For each positive integer n we define

$$\Gamma_n = u^n(\beta_1)^n \qquad b_n = u^n \beta_n \tag{1.1}$$

where $u \in \pi_2(K)$ is the usual generator and the product $(\beta_1)^n$ is induced by the tensor product : $CP^\infty \times CP^\infty \to CP^\infty$. Now using the result of [4], [5] one can prove (see [1] for details) that

$$\Gamma n = \sum_{r=1} r!\, S_n^r\, b_r \tag{1.2}$$

were S_n^r is the sterling number of the second kind.

Let $i: CP^\infty \to BU$ be the canonical inclusion and denote the images of β_n , Γ_n under i_* also by β_n, Γ_n respectively. The later classes of course can be multiplied in $K_*(BU)$ by using the Whitney sum maps: $BU(m) \times BU(n) \to BU(m + n)$. The following is a well known (see[6; p.47] or [7; 16.31]).

Theorem 1.3

(i) $K_*(BU(n))$ is free over $\pi_*(K)$ with a base consisting of the monomials

$$\beta_{i_1} \beta_{i_2} \cdots \beta_{i_r}$$

such that $i_1 > 0, i_2 > 0, \ldots , i_r\ 0, 0 \le r \le n$ (The monomial with $r = 0$ is interpreted as 1)

(ii) $K_*(MU(n))$ is free over $\pi_*(K)$ with a base consisting of the monomials

$$\beta_{i_1} \beta_{i_2} \cdots \beta_{i_r}$$

such that $i_1 > 0, i_2 > 0, \ldots i_n > 0$.

(iii) $K_*(BU)$ is the polynomial algebra $\pi_*(K) [\beta_1 , \beta_2, \ldots , \beta_n, \ldots]$.

Remark 1.4

If we replace the β_i's by the b_i's then all the statements given in the above theorem remain true.

§2. Primitivity in $K_*(BU)$

In this section we shall see that the classes $\Gamma_1 , \Gamma_2, \ldots$ play an important role in the determination of the primitive clases of $K_*(BU)$.

Consider the Hurewicz homomorphism $h_K^s : \pi_*^s (BU) \to K_*(BU)$. It is easy to see that Γ_1 is in the image of h_K^s . But this is a natural homomorphism of Pontrjagin rings, hence it follows that all the classes Γ_1 , Γ_2 , ... are in the image of h_K^s and hence

$$\text{Im } h_K^s \supseteq \mathbb{Z}\,[\Gamma_1, \Gamma_2, \ldots , \Gamma_n , \ldots] \qquad 2.1$$

Now since $K_*(BU)$ is torsion-free, the map: $K_*(BU) \to K_*(BU) \otimes \mathbb{Q}$ is a monomorphism. By denoting the image of Γ_n under this map also by Γ_n we have

$$\text{Im } h_{K\mathbb{Q}}^s \supseteq \mathbb{Q}\,[\Gamma_1, \Gamma_2, \ldots , \Gamma_n , \ldots]$$

where $h_{K\mathbb{Q}}^s : \pi_*^s (BU) \otimes \mathbb{Q} \to K_*(BU) \otimes \mathbb{Q} = (K\mathbb{Q})_* (BU)$.

In fact by a simple application of the Atiyah-Hirzebruch spectral sequence

$$E^2_{u,v} = \tilde{H}_u (BU; \pi_v^s) \Rightarrow \pi_{u+v}^s(BU)$$

one can prove by comparing the ranks that (see [1] for details)

$$\text{Im } h_{K\mathbb{Q}}^s = \mathbb{Q}\,[\Gamma_1 , \Gamma_2 , \ldots , \Gamma_n , \ldots] \qquad (2.2)$$

Now since $\pi_*(K)$ is torsion-free $h_{K\mathbb{Q}}^s$ maps $\pi_*^s (BU) \otimes \mathbb{Q}$ isomorphically onto $P(K\mathbb{Q})_*$ (BU). Hence we have proved the following:

Proposition 2.3

$$P(K\mathbb{Q})_* (BU) \approx \mathbb{Q}\,[\Gamma_1 , \Gamma_2 , \ldots , \Gamma_n , \ldots]$$

The following result can be proved exactly the same as (2.3) or alternatively one can use the above result and the stable equivalence $BU = \bigvee_{n=1}^{\infty} MU(n)$ of [8; Th. 1.4.2] to prove it.

Proposition 2.4

$P(K\mathbb{Q})_* (MU(n))$ is free over $\mathbb{Q}$ with a base consisting of all monomials $\Gamma_{i_1} \Gamma_{i_2} \ldots \Gamma_{i_n}$ such that $i_1 > 0, i_2 > 0, \ldots , i_n > 0$.

Note that such a monomial is not divisible in $K_*(MU(n))$, but more complicated expressions may be well be. In fact we have the following (see [1 ; 6.32]).

Theorem 2.5

An element A in $K_*(MU(n)$ is primitive if and only if it can be written in the form

$$A = \sum_i \lambda_i \Gamma_{m_{i,1}} \Gamma_{m_{i,2}} \ldots \Gamma_{m_{i,n}} \qquad \lambda_i \in \mathbb{Q}$$

such that when we rewrite it in terms of the $\pi_*(K)$-base $\{ b_{i_1} b_{i_2} \ldots b_{i_n} \}$, the induced formula has integral coefficients.

Proof

Since $K_*(MU(n))$ is torsion-free we have a monomorphism α: $K_*(MU(n)) \to K_*(MU$ $(n)) \otimes \mathbb{Q} \approx (K\mathbb{Q})_* (MU(n))$. Now let

$$A = \sum_i \lambda_i \Gamma_{m_{i,1}} \Gamma_{m_{i,2}} \dots \Gamma_{m_{i,n}} \qquad \lambda_i \in \mathrm{Q}$$

Then by the above theorem A is in $P(K\mathrm{Q})_*$ $(MU(n))$. Now if A also satisfies the condition of the theorem then it is in the image of α and so it represents an element of PK_* $(MU(n))$ as required.

Conversely if A is in $PK_*(MU(n))$ then it is also in $P(K\mathrm{Q})_*$ $(MU(n))$ where we identify A with its image under the monomorphism α. Hence by (2.4) we can write A in the form

$$A = \sum_i \lambda_i \Gamma_{m_{i,1}} \Gamma_{m_{i,2}} \dots \Gamma_{m_{i,n}} \qquad \lambda_i \in \mathrm{Q}$$

Note that the condition of the theorem is satisfied since A essentially is in $K_*(MU(n))$.

Theorem 2.6

An element A *in* $K_*(MU(n))$ is primitive if and only if it can be written in the form

$$A = \sum_i \lambda_i \Gamma_{m_{i,1}} \Gamma_{m_{i,2}} \dots \Gamma_{m_{i,n}} \qquad \lambda_i \in \mathrm{Q}$$

such that $\lambda_1, \lambda_2, \dots, \lambda_k$ are rational numbers satisfy the following condition

$$\frac{1}{n_1!n_2!....n_r!} \sum_i \lambda_i \left(\left[\sum_\varphi k_{\varphi(1)}^{m_{i,1}} \; k_{\varphi(2)}^{m_{1,2}} \; \dots \; k_{\varphi(n)}^{m_{i,n}} \right] \right)$$

is an integer for all n-tuples $(k_1, k_2, \dots k_n)$ of positive integers contains r- distinct elements repeated $n_1, n_2, \dots n_r$ times, respectively, where φ runs over all the permutations of $(1, 2, \dots n)$.

Proof

Suppose that A is a primitive class of $K_*(MU(n))$. Then by the above theorem we can write it in the form

$$A = \sum_i \lambda_i \Gamma_{m_{i,1}} \Gamma_{m_{i,2}} \dots \Gamma_{m_{i,n}} \qquad \lambda_i \in \mathrm{Q}$$

such that when we write it in terms of the $\pi_*(K)$-base $\{b_{i_1} b_{i_2} \dots b_{i_n}\}$, the induced formula has integral coefficients. Now by (1.2) we have

$$A = \sum_i \lambda_i \prod_{j=1}^{n} \left(\sum_{r_i, j=1} (r_{i,j})! \; S_{m_{i,j}}^{r_{i,j}} b_{r_{i,j}} \right) .$$

Let $a_{(k_1, k_2, \dots, k_n)}$ be the coefficient of $b_{k_1} b_{k_2} \dots b_{k_n}$. Then we have

$$a(k_1, k_2, \ldots, k_n) = \frac{1}{n_1! n_2! \ldots n_r!} \sum_i \lambda_i \left\{ \sum_{\varphi} \prod_{j=1}^{n} \left((k_{\varphi(j)}! \ S_{m_{i,j}}^{k_{\varphi(j)}}) \right) \right\} .$$

To complete the long and tedious proof of the theorem one use the induction on $k = k_1 + k_2 + \ldots + k_n$ together with the formula (see [9; p. 226]).

$$r! S_n^r = \sum_{t=0} (-1)^t \binom{r}{t} (r-t)^n$$

See [1; 6.33] for the proof of the two dimensional case.

Remark 2.7. As we mentioned before the primitive classes of $K_*(BU(n)$ are represented in [3] by rational polynomials satisfy certain conditions. The primitive class A in the above theorem is corresponding to the rational polynomial $f(x_1, \ldots, x_n)$ defined by

$$f(x_1, \ldots, x_n) = \frac{1}{n!} \sum_i \lambda_1 \left(\sum_{\varphi} x_{\varphi(2)}^{m_{i,1}} \ x_{\varphi(2)}^{m_{i,2}} \ldots x_{\varphi(n)}^{m_{i,n}} \right)$$

§3. Some Primitive Elements in $K_*(BU)$

Here we shall use theorem (2.6) to determine the primitive classes of $K_*(MU(n))$ of the form $\lambda(\Gamma_m^{n-1} \Gamma_{m+ns} - G_{m+s}^n)$, where $\lambda \in Q$.

By the above theorem such a class is primitive if and only if the following expression

$$\lambda = \frac{(n-1)!}{n_1! n_2! \ldots n_r!} (k_1 k_2 \ldots k_n)^m \left[\sum_{i=1}^{n} k_i^{ns} - n(k_1 k_2 \ldots k_n)^s \right]$$

is an integer for each n-tuple $(k_1, k_2, \ldots k_n)$ of positive integers containing r distinct elements repeated $n_1, n_2, \ldots, n_r$ times, respectively.

Notations 3.1. Let

(i) $$X_{(k_1, k_2, \ldots, k_n)} = \left(\sum_{i=1}^{n} k_i^{ns} \right) - n(k_1 k_2 \ldots k_n)^s ,$$

(ii) $$X_k = X_{(k,1, \ldots, 1)}$$

It is easy to show that

Proposition 3.2

(i) $$X_{(k_1, k_2, \dots, k_n)} = \sum_{i=1}^{n} X_{k_i} - n\sum_{i-2}^{n} \left[(k_1k_2...k_{i-1})^s - 1\right](k_i^s - 1)$$

(ii) $$X_k = (k^s - 1)^2 \left[(k^{s(n-2)} - 1) + 2(k^{s(n-3)} - 1) + ... + (n-2)\ (k^s - 1) + \frac{n(n-1)}{2} \right]$$

Next we recall the definition of a numerical function $m(t)$ defined on the positive integers. Let $v_p(k)$ be the exponent of the prime p in k, so that $k = \Pi_p\, p^{v_p(k)}$.

Definition 3.3. [10] If t is a positive integer, we define $m(t)$ by

$$v_2(m(t)) = \begin{cases} 1 & t \text{ odd} \\ 2 + v_2(t) & t \text{ even} \end{cases}$$

$$\text{for } p \text{ odd } v_p(m(t)) = \begin{cases} 1 & p-1 \text{ not divide } t \\ 1 + v_p(t) & p-1 \text{ divides } t \end{cases}$$

Let $M_n(t)$ be the highest common factor of the expressions $k^n(k^t – 1)$ where k runs over the positive integers. One can prove the following

Proposition 3.4.

[10; p. 143] For each prime p we have,

$$v_p(M_n(t) = \text{Min } \{ n , v_p (m(t)) \}$$

In particular when n is large enough we have $Mn(t) = m(t)$.

Returning to our case we want to find the divisibility in the expression

$$\frac{(n-1)!}{n_1!n_2!...n_r!} \ (k_1k_2...k_n)^m \ \ X_{(k_1, k_2, ..., k_n)} \qquad (*)$$

From 3.2 (ii) it is easy to show that when n is an odd prime we can choose s (may take $s = n – 1$) such k^mX_k is a multiplication of $M_m(n, s)$ where we define

$$v_p(M_m(n, s)) = \text{Min } \{ m, v_p (n.m^2 (s)) \} \qquad (3.5)$$

Note that when m is big enough we have $M_m(n, s) = n.m^2(s)$.

Now n is an odd prime. Hence $(n – 1)! / n_1n_2! ... n_r!$ is an integer. Therefore it follows from the above remark together with fomula (i) of (3.2) that the expression (*) is divisible by $M_m(n,s)$. We have proved the following:

Proposition 3.6.

Let p be an odd prime. For each positive integer m there is a primitive class in $K_*(MU(p))$ of the form

$$\frac{\Gamma_m^{p-1}\,\Gamma_{m+p(p-1)} - \Gamma_{m+p-1}^{p}}{M_m(p, p-1)}$$

Finally we want to mention that the primitive classes constructed here are only examples of the use of theorem (2.6) and we can form more of them using the same method. Also all of these classes can be constructed in the language of [3] using the same method (see remark (2.7)), but here the primitive classes are more recognizable.

References

[1] **Felali, K.S.,** *Intersection points of immersed manifolds,* Thesis, Univ. of Manchester, 1982.

[2] **Ray, N.** and **Schwartz, L.,** Construction d´ éléments dans $\pi_*^S(BU(2))$, *Bull. Soc. Math. France* **111** (1983), 449-465.

[3] **Baker, A., Clarke, F., Ray, N.** and **Schwartz, L.,** On the Kummer congruences and the stable homotopy of BU, *Trans. American Math. Sco.,* **316** (1989), no. 2, 385-432.

[4] **Clarke, F.,** Self-map of *BU, Math. Proc. Camb. Phil. Soc.,* **89** (1981), 491-500.

[5] **Schwartz, I.,** Opérations d´ Adams en *K*-homologie et applications, *Bull. Soc. Math. France,* **109** (1981) 237-257.

[6] **Adams, J.F.,** *Stable Homotopy and Generalized Homology,* Chicago Lectures Notes in Math. (1974).

[7] **Switzer, R.M.,** *Algebric Topology-Homotopy and Homology,* Grudlehren Math. Vol. **212,** Springer-Verlag (1975).

[8] **Snaith, V.,** *Algebraic Cobordism and K-theory,* Memoris A.M.S. vol. **21** (1979).

[9] **Riordan, J.,** *Combinatorial Identities,* Wiley, New York (1968).

[10] **Adams, J.F.,** On the group *J*(*X*) – II, *Topology* **3** (1965), 137-171.

الفصـــول الجـــذريـة للنظم الجبريـــة $K_*(BU)$

خالد فيلالي

قسم العلوم الرياضية ، جامعة أم القرى

مـــكة المكرمـــة - المملكة العربية السعودية

المستخلص . الفصول الجذرية للنظم الجبرية $K_*(BU(n))$ للفراغات $BU(n)$ بدأت دراستها منذ عام ١٩٨٠م إلا أن جميع الدراسات السابقة تركزت على الحالة الخاصة عندما $n = 2$.

في هذا البحث نحدد بعض الفصول الجذرية لنظم تكون فيها $n \geq 3$.