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Symmetric Generating Set of the Groups $A_{kn\,+}$ and $S_{kn\,+\,1}$

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ABSTRACT. In this paper we will show how to generate in general A_{kn+1} and S_{kn+1} – the alternating and the symmetric groups of degrees $kn + 1$ – using a copy of S_n and an element of order $k + 1$ in A_{kn+1} and S_{kn+1} for all positive integers $n \geq 2$ and $k \geq 2$. We will also show how to generate A_{kn+1} and S_{kn+1} symmetrically using n elements each of order $k + 1$.

I. Introduction

Hammas[1], showed that A_{2n+1} can be presented as

$$G = A_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n,\ T^3 = [\, T, S_{s-1} \,] = 1 \rangle$$

for all $3 \leq n \leq 11$ where $[\, T, S_{n-1} \,]$ means that T commutes with Y and with $(XY)^{X^{-(n-3)}}$, the generators of S_{n-1}. The relations of the symmetric group $S_n = \langle X, Y \rangle$ of degree n are found in Coxeter and Moser[3]. In order to complete the enumeration, we need to add some relations to the presentation that generate A_{2n+1}, $n \geq 2$. Also in Hammas[1] it has been proved that for all $3 \leq n \leq 11$, the group A_{2n+1} can be symmetrically generated by n-elements each of order 3, and of the form $T_0, T_1, \dots, T_{n-1}$, where, $T_i = T^{X^i} = X^{-i}\, TX^i$ and T, X satisfy the relations of the group A_{2n+1}. The set $\{ T_0, T_1, \dots, T_{n-1} \}$ is called the symmetric generating set of A_{2n+1} (see the definition in section 2).

Hammas *et al.*[2] have given a permutational generating set that generates A_{2n+1} for all $n \geq 2$, and satisfies the relations given in the group G above. Also, they have proved that for all $n \geq 2$, the group A_{2n+1} can be symmetrically generated by n-elements each of order 3[2].

In this paper, we give permutations generate A_{kn+1} and S_{kn+1} for all $n \geq 2$ and satisfy the relations given in presentation of A_{kn+1} and S_{kn+1}. Further, we prove that A_{kn+1} and $S_{kn+1}: n \geq 2$ can be symmetrically generated by n permutations each of order 3 satisfying our definition in Hammas *et al.*[2].

The results obtained here generalise the results given Hammas *et al.*[2] and lead us to formulate a conjecture which generalises the results given in Hammas[1].

II. Symmetric Generating Sets

Let G be a group and $\Gamma = \{ T_0, T_1, \ldots, T_{n-1} \}$ be a subset of G where, $T_i = T^{x^i}$ for all $i = 0, 1, \ldots, n-1$. Let S_n a copy of the symmetric group of degree-n be the normalizer in G of the set Γ. We define Γ to be a symmetric generating set of G if and only if $G = \langle \Gamma \rangle$ and S_n permutes Γ doubly transitively by conjugation, i.e., Γ is realizable as an inner automorphism.

III. Permutational Generating Set of A_{kn+1} and S_{kn+1}

Theorem III.1. A_{kn+1} and S_{kn+1} can be generated using a copy of S_n and an element of order $k+1$ in A_{kn+1} and S_{kn+1} for all $n \geq 2$ and all $k \geq 2$.

Proof

Let X, Y and T be the permutations :
$X = (1, 2, \ldots, n)(n+1, n+2, \ldots, 2n) \ldots ((k-1)n+1, (k-1)n+2, \ldots, kn)$,
$Y = (1, 2)(n+1, n+2) \ldots ((k-1)n+1, (k-1)n+2)$, and $T = (n, 2, n, 3n, \ldots, kn, kn+1)$
be three permutations; the first of order n, the second of order 2 and the third of order $k+1$. Let H be the group generated by X and Y. By Coxeter and Moser[3], the group H is the symmetric group S_n. Let $\bar{G}$ be the group generated by X, Y and T. Consider the product TX. Let $\beta = (TX)^n$. Let $K = \langle \beta, T \rangle$. Since

$\beta = (1, n+1, 2n+1, 3n+1, \ldots, kn+1, n, 2n, 3n, \ldots kn, n-1, 2n-1, \ldots kn-1, n-2, 2n-2, \ldots, kn-2, \ldots, \ldots, 2, n+2, 2n+2, \ldots, (k-1)n+2)$

then we claim that K is either A_{kn+1} or S_{kn+1}. To show this, let θ be the mapping which takes the element in the position i of the permutation β into the element i in the permutation $(1, 2, \ldots, kn+1)$. Under this mapping θ, the group K will be mapped into the group

$$\theta(K) = \langle (1, 2, \ldots, kn+1), (n-1, n, n+1, \ldots, n+k) \rangle.$$

Now if k is an odd integer then $(n-1, n, n+1, \ldots, n+k)$ is an odd permutation. Hence $\theta(K)$ is the symmetric group S_{kn+1}. Since K is a subgroup of $\bar{G}$, then $\bar{G}$ is the symmetric group S_{kn+1}. While if k is an even integer then the permutations $(1, 2, \ldots, kn+1)$ and $(n-1, n, n+1, \ldots, n+k)$ are even. Hence $\theta(K)$ is the alternating group A_{kn+1}. In this case X, Y and T are all even permutations. Therefore $\bar{G}$ is A_{kn+1}. ◇.

Conjecture

The above theorem led us to state the following conjecture which generalizes the result proved by Hammas[1]

Let $G = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^{k+1} = [T, S_{n-1}] =$

for all $n \geq 2$ and all $k \geq 2$. If k is an even integer when $G \cong S_{kn+1}$.

It is important to notice that the elements X, Y and T described above satisfy the relations of the group G given in the conjecture above. In particular, the elements X, Y generate a copy of S_n. The elements Y and T commute for all $n \geq 3$. For all $n \geq 3$, the element T commutes with the group $S_{n-1} = \langle Y, (XY)^{X^{-(n-3)}} \rangle$.

IV. Symmetric Permutational Generating Set of A_{kn+1} and S_{kn+1}

Theorem IV.1. Let

$X = (1, 2, \ldots, n)(n+1, n+2, \ldots, 2n) \ldots ((k-1)n+1, (k-1)n+2, \ldots, kn)$, $Y = (1,2)(n+1, n+2) \ldots ((k-1)n+1, (k-1)n+2)$ and $T = (n, 2, n, 3n, \ldots, kn, kn+1)$ be the permutations described before. Let $\Gamma = \{ T_0, T_1, \ldots, T_{n-1} \}$ for all $n \geq 2$, where $T_i = T^{X^i}$. *If* k is an even integer, then the set Γ generates the alternating group A_{kn+1} symmetrically. If k is an odd integer, then the set Γ generates the symmetric group S_{kn+1} symmetrically.

Proof

Let $T_0 = (n, 2n, \ldots, kn, kn+1)$, $T_1 = T^X = (1, n+1, \ldots, (k-1)n+1, kn+1), \ldots, T_{n-1} = T^{X^{n-1}} = (n-1, 2n-1, 3n-1, \ldots, kn-1, kn+1)$. Let $H = \langle \Gamma \rangle$. We claim that $H \cong A_{kn+1}$ or S_{kn+1}. To show this, consider the element.

$$\alpha = \prod_{i=}^{n} T^{X^i}$$

It is not difficult to show that

$\alpha = (1, n+1, 2n+1, 3n+1, \ldots, (k-1)n+1, 2, n+2, 2n+2, \ldots, (k-1)n+2, \ldots, n, 2n, 3n, \ldots, kn, kn+1)$.

Let $H_1 = \langle \alpha, T_0 \rangle$. We claim that $H_1 \cong H_{kn+1}$ or S_{kn+1}. To prove this, let θ be the mapping which takes the element in the position i of the cycle α into the element i of the cycle $(1, 2, \ldots, kn+1)$. Under this mapping the group H_1 will be mapped into the group

$\theta(H_1) = \langle (1, 2, \ldots, kn+1), (k(n-1)+1, k(n-1)+2, \ldots, kn, kn+1) \rangle$.

As in the proof of the previous theorem we can conclude that if k is an odd integer then $H \cong H_1 \cong \theta(H_1) \cong S_{kn+1}$, and if k is an even integer then $H \cong H_1 \cong \theta(H_1) \cong A_{kn+1}$. ◇

The set Γ described above satisfies the conditions of the group given in Hammas[1]. It is important to note that Γ has to have at least n elements each of order $k+1$ to generate A_{kn+1} or S_{kn+1}. The following theorem characterizes all groups found if we remove m-elements of the set Γ.

Theorem IV.2 Let T and X be the permutations which have been described above, where $T^{K+1} = 1$. Let $\Gamma = \{ T_1, T_2, \ldots, T_n \}$ for all $n \geq 2$, where $T_i = T^{X^i}$. If k is an even integer then if we remove m-elements of the set Γ for all $1 \leq m \leq n-2$ then the resulting set generates $A_{k(n-m)+1}$. If k is an odd integer then if we remove m-elements of the set Γ for all

$1 \leq m \leq n-2$ then the resulting set generates $S_{k(n-m)+}$. If we remove $(n-1)$ – elements of the set Γ then the resulting set generates C_{k+1}.

Proof

Using induction on $n-m$, if $n-m=1$ then let $\Gamma_1 = \{T_1\}$. Since T_1 is the permutation $(1, n+1, \dots, (k-1)n+1, kn+1)$ of order $k+1$ then Γ_1 generates C_{k+1}. Suppose that $1 \leq m \leq n-2$. Assume that the theorem is true for $n-m=j$. i.e., if $\Gamma_j = \{T_1, \dots, T_j\}$ then Γ_j generates $A_{k(j)+1}$ or $S_{k(j)+1}$ depending on whether k is an even or an odd integer respectively. For $n-m=j+1$, let $\Gamma_{j+1} = \{T_1, \dots, T_{j+1}\}$. Let $F = \{T_1, \dots, T_j\}$. By this hypothesis, F generates $A_{k(j)+1}$ or $S_{k(j)+1}$. Since

$\beta = (1, n+1, 2n+1, 3n+1, \dots, (k-1)n+1, 2, n+2, 2n+2, \dots, (k-1)n+2, \dots, j, n+j, 2n+j, \dots, (k-1)n+j, kn+1) \in \langle F \rangle$,

and since $T_{j+1} = T^{x^{j+1}} = (j+1, n+j+1, 2n+j+1, \dots, (k-1)n+j+1, kn+1)$ then

$\beta T_{j+1} = (1, n+1, 2n+1, \dots, (k-1)n+1, 2, n+2, 2n+2, \dots, (k-1)n+2, \dots, j, n+j, 2n+j, \dots, (k-1)n+j, j+1, n+j+1, 2n+j+1, \dots, (k-1)n+j+1, kn+1) \in \langle F, T_{j+1} \rangle$.

But $\langle F, T_{j+1} \rangle \cong A_{k(j)+1}$ or $S_{k(j)+1}$ depending on whether k is an even or an odd integer respectively, and so the theorem is true for all m. ◇

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مجموعـــة المولـــدات المتمـــاثلة للزمـــر A_{kn+1} و S_{kn+1}

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المستخلص . في هذا البحث نقدم كيف يمكن توليد زمر التناظرات من الدرجة $kn+1$ بشكل عام باستخدام صورة من زمرة التناظرات S_n وعنصر من الرتبة $k+1$ في الزمرة A_{kn+1} و الزمرة S_{kn+1} لكل الأعداد الصحيحة الموجبة $n \geq 2$ و $k \geq 2$. كذلك سوف نقدم برهانا يثبت أن الزمر A_{kn+1} و S_{kn+1} يمكن توليدها باستخدام مجموعة مولدات التماثل التي تتكون من عدد n من العناصر ذات الرتبة $k+1$.